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SINGULAR PERTURBATIONS OF SEMILINEAR SECOND ORDER SYSTEMS. (U)

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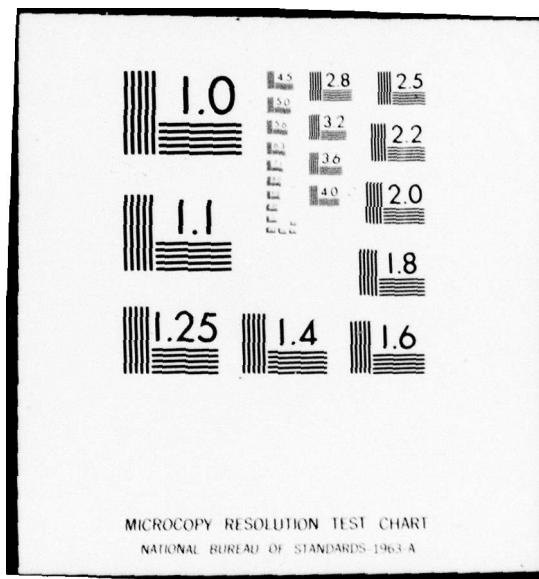
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SINGULAR PERTURBATIONS OF SEMILINEAR SECOND ORDER SYSTEMS

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FOR FURTHER TRAN



1. Problems with boundary layers at one endpoint

Many physical problems can be studied as singularly perturbed two-point vector boundary value problems of the form

$$(1) \quad \begin{cases} \epsilon y'' + f(y, t, \epsilon)y' + g(y, t, \epsilon) = 0, & 0 \leq t \leq 1 \\ y(0), y(1) \text{ prescribed} \end{cases}$$

where ϵ is a small positive parameter (cf., e.g., Amundson (1974), Sethna and Balachandra (1976), and Cohen (1977)). Scalar problems of this form are analyzed quite thoroughly in the forthcoming memoir, Howes (1978). An enlightening case history of such analyses was given by Erdélyi (1975), and important early work includes that of Coddington and Levinson (1952) and Wasow (1956).

For simplicity, let us assume that f and g are infinitely differentiable in y and t and that they possess asymptotic power series expansions in ϵ as $\epsilon \rightarrow 0$. We'll first consider the vector problem under the assumption that the reduced problem

$$(2) \quad f(u_R, t, 0)u'_R + g(u_R, t, 0) = 0, \quad u_R(1) = y(1)$$

is stable throughout $0 \leq t \leq 1$ in the sense that u_R exists and

$$(3) \quad f(u_R(t), t, 0) > 0$$

there (i.e., $-f$ is a strictly stable matrix having eigenvalues with negative

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* real parts). We first realize that u_R cannot generally represent the solution to (1) near $t = 0$ because we cannot expect to have $u_R(0) = y(0)$. Instead, we must expect boundary layer behavior to occur near $t = 0$, providing the required nonuniform convergence from $y(0)$ to $u_R(0)$ as $\epsilon \rightarrow 0$. For a (very) small "boundary layer jump" $\|y(0) - u_R(0)\|$ or for a constant $f(y, 0, 0)$, no extra hypotheses are needed. More generally, however, we must require an additional "boundary layer stability" assumption, namely that the inner product

$$(4) \quad \zeta^T \int_0^{\zeta} f(u_R(0)+s, 0, 0) ds > 0$$

remains positive for $\zeta + u_R(0)$ along all paths connecting $u_R(0)$ and $y(0)$ with $0 < \|\zeta\| \leq \|y(0) - u_R(0)\|$. (Here, T represents the transpose and $\|z\| = \sqrt{z^T z}$.) We note that if $f(z, 0, 0)$ is the gradient $\nabla F(z - u_R(0))$, (4) is equivalent to the condition that

$$\zeta^T (F(\zeta) - F(0)) > 0$$

since the integral is then path-independent. Indeed, (4) directly generalizes the (minimal) hypotheses used by Howes for the scalar problem and it is weaker than the common assumption that $f(y,0,0) > 0$ for all y .

Pictorially, the boundary layer stability assumption must hold within the circle shown

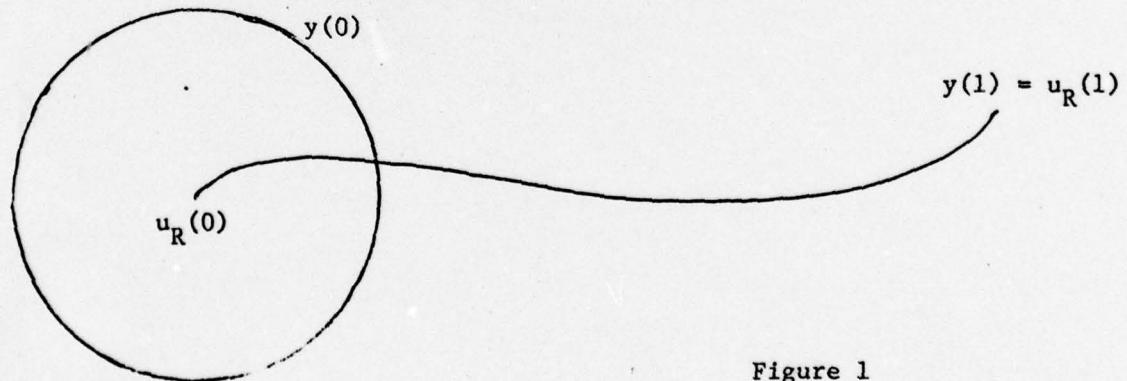


Figure 1

The results of Howes and others suggest that under such hypotheses, (1) will have a solution $y(t, \epsilon)$ of the form

$$(5) \quad y(t, \varepsilon) = U(t, \varepsilon) + \Pi(t, \varepsilon)$$

where the outer solution U has an asymptotic expansion

$$(6) \quad U(t, \epsilon) \sim \sum_{j=0}^{\infty} U_j(t) \epsilon^j$$

providing the asymptotic solution for $t > 0$, while the boundary layer correction Π has an expansion

$$(7) \quad \Pi(\tau, \epsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau) \epsilon^j$$

whose terms all tend to zero as the stretched variable

$$(8) \quad \tau = t/\epsilon$$

tends to infinity. We would expect this solution to be unique. Under weaker smoothness assumptions on f and g , we'd have to limit the expansions to finite order approximations. For the scalar problem, Howes doesn't actually obtain higher order terms or complete boundary layer behavior, but they can easily be generated. Applying his results to the boundary value problem for the remainder terms, however, shows the asymptotic validity of the expansions so obtained.

The outer expansion (6) must provide the asymptotic solution to (1) for $t > 0$, since Π is then asymptotically negligible. Thus, the terms U_j can be successively obtained by equating coefficients in the terminal value problem

$$(9) \quad f(U(t, \epsilon), t, \epsilon)U'(t, \epsilon) + g(U(t, \epsilon), t, \epsilon) = -\epsilon U''(t, \epsilon), \quad U(1, \epsilon) = y(1).$$

Evaluating at $\epsilon = 0$, then, shows that U_0 must satisfy the reduced problem

$$(10) \quad f(U_0(t), t, 0)U'_0(t) + g(U_0(t), t, 0) = 0, \quad U_0(1) = y(1)$$

(which has a unique solution $u_R(t)$ under (2) and (3)). Succeeding terms U_j , $j > 0$, will satisfy linear problems of the form

$$(11) \quad f(U_0(t), t, 0)U'_j(t) + f_y(U_0(t), t, 0)U_j(t)U'_0(t) + g_y(U_0(t), t, 0)U_j(t) = h_{j-1}(t), \quad U_j(1) = 0$$

where h_{j-1} is known in terms of $t, U_0(t), \dots, U_{j-1}(t)$. The stability assumption (3) implies that (11) is a nonsingular initial value problem, so it also has a unique solution throughout $0 \leq t \leq 1$. Thus, there is no difficulty in generating the outer expansion $U(t, \epsilon)$ with $U(t, 0) = u_R(t)$.

The boundary layer correction Π must necessarily be a decaying solution of the nonlinear initial value problem

$$(12) \quad \left\{ \frac{d^2\Pi}{d\tau^2} + f(U(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon\tau, \epsilon) \frac{d\Pi}{d\tau} = -\epsilon[(f(U(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon\tau, \epsilon) -$$

$$(12) \quad \left\{ \begin{array}{l} - f(U(\varepsilon\tau, \varepsilon), \varepsilon\tau, \varepsilon)) \frac{dU}{d\tau}(\varepsilon\tau, \varepsilon) + g(U(\varepsilon\tau, \varepsilon) + \Pi(\tau, \varepsilon), \varepsilon\tau, \varepsilon) \\ - g(U(\varepsilon\tau, \varepsilon), \varepsilon\tau, \varepsilon)], \quad \tau \geq 0 \\ \Pi(0, \varepsilon) = y(0) - U(0, \varepsilon). \end{array} \right.$$

Thus, the leading term Π_0 must satisfy the nonlinear problem

$$(13) \quad \frac{d^2\Pi_0}{d\tau^2} + f(U_0(0) + \Pi_0(\tau), 0, 0) \frac{d\Pi_0}{d\tau} = 0, \quad \Pi_0(0) = y(0) - U_0(0)$$

while later terms must satisfy linear problems

$$(14) \quad \begin{aligned} & \frac{d^2\Pi_j}{d\tau^2} + f(U_0(0) + \Pi_0(\tau), 0, 0) \frac{d\Pi_j}{d\tau} \\ & + f_y(U_0(0) + \Pi_0(\tau), 0, 0) \Pi_j(\tau) \frac{d\Pi_0}{d\tau} = k_{j-1}(\tau), \quad \Pi_j(0) = -U_j(0) \end{aligned}$$

where k_{j-1} is a linear combination of preceding terms Π_ℓ and their derivatives $d\Pi_\ell/d\tau$, $\ell < j$, with coefficients that are functions of τ and $\Pi_0(\tau)$. The decaying solution of (13) must satisfy

$$\frac{d\Pi_0}{d\tau} + \int_{-\infty}^{\tau} f(U_0(0) + \Pi_0, 0, 0) \frac{d\Pi_0}{d\tau} d\tau = 0$$

and, thereby, the initial value problem

$$(15) \quad \frac{d\Pi_0}{d\tau} = - \int_0^{\Pi_0(\tau)} f(U_0(0) + w, 0, 0) dw, \quad \tau \geq 0, \quad \Pi_0(0) = y(0) - u_R(0).$$

Multiplying by Π_0^T , the boundary layer stability condition (4) implies that

$$(16) \quad \frac{1}{2} \frac{d}{d\tau} \|\Pi_0(\tau)\|^2 = -\Pi_0^T(\tau) \int_0^{\Pi_0(\tau)} f(U_0(0) + z, 0, 0) dz < 0$$

for nonzero values of $\Pi_0(\tau)$ satisfying $\|\Pi_0(\tau)\| \leq \|y(0) - u_R(0)\| = \|\Pi_0(0)\|$. Thus, our boundary layer stability implies that $\|\Pi_0(\tau)\|$ will decrease monotonically as τ increases until we reach the rest point $\Pi_0(\tau) = 0$ of (15) at $\tau = \infty$. Ultimately, $\Pi_0(\tau)$ will become so small that (3) (for $t = 0$) implies that the eigenvalues of $f(U_0(0) + \Pi_0(\tau), 0, 0)$ will thereafter have real parts greater than some $\kappa > 0$ and (15) then implies that

$$(17) \quad \Pi_0(\tau) = 0(e^{-\kappa\tau}),$$

i.e., Π_0 is exponentially decaying as $\tau \rightarrow \infty$. Although we can seldom explicitly integrate the nonlinear system (15), we can approximate its solution arbitrarily closely by using a successive approximations procedure on (15) (cf. Erdélyi (1964)). Knowing Π_0 , we next integrate (14) for $j = 1$ and then proceed term-wise. Rearranging (14) and integrating, we obtain

$$\frac{d\Pi_1}{d\tau} + f(u_0(0) + \Pi_0(\tau), 0, 0)\Pi_1 + \ell_1(\tau) = 0$$

where

$$\begin{aligned} \ell_1(\tau) \equiv & \int_{-\infty}^{\tau} \{ f_y(u_0(0) + \Pi_0(r), 0, 0) [\Pi_1(r) \frac{d\Pi_0}{d\tau}(r) \\ & - \frac{d\Pi_0}{d\tau}(r)\Pi_1(r)] + k_{j-1}(r) \} dr \end{aligned}$$

is known whenever Π_1 and $d\Pi_0/d\tau$ commute. Thus, Π_1 satisfies the integral equation

$$(18) \quad \Pi_1(\tau) = P(\tau)u_1(0) - \int_0^{\tau} P(\tau)P^{-1}(r)\ell_1(r)dr$$

where $P(\tau)$ is the exponentially decaying fundamental matrix for the linear system

$$\frac{d\Pi}{d\tau} + f(u_0(0) + \Pi, 0, 0)\Pi = 0, \quad \tau \geq 0, \quad \Pi(0) = I.$$

In general, (18) must also be solved via successive approximations, though it directly provides the solution of (14) when the commutator $[\Pi_1, d\Pi_0/d\tau] \equiv 0$.

We note that the boundary layer jump $\|\Pi_0(0)\| = \|y(0) - u_R(0)\|$ is limited by the minimum value of $\|\zeta\| > 0$ such that the inner product

$$(19) \quad \zeta^T \int_0^{\zeta} f(u_R(0) + z, 0, 0) dz = 0.$$

The jump can, in practice, be quite large. It involves no restriction, for example, if $f(z, 0, 0) > 0$ for all z since then (19) cannot ever hold for a $\zeta \neq 0$. It gives precise limits to the jumps for certain scalar problems (Howes (1977) reconsiders an example of O'Malley (1974)).

We could also consider the reduced problem

$$(20) \quad f(u_L, t, 0)u'_L + g(u_L, t, 0) = 0, \quad u_L(0) = y(0).$$

Then the stability condition (3) and the boundary layer stability condition (4) would be replaced by

$$(21) \quad f(u_L(t), t, 0) < 0$$

for $0 \leq t \leq 1$ and the assumption that

$$(22) \quad \theta^T \int_0^\theta f(u_L(1) + z, 1, 0) dz < 0$$

for all $\theta + u_L(1)$ on paths between $u_L(1)$ and $y(1)$ satisfying $0 < \|\theta\| \leq \|y(1) - u_L(1)\|$. Nonuniform convergence of the solution to (1) would then take place near $t = 1$, depending on the stretched variable

$$\sigma = (1 - t)/\epsilon,$$

and the limiting solution on $0 \leq t < 1$ would be $u_L(t)$. If f were nonsingular with eigenvalues having both positive and negative real parts along an appropriate solution of the reduced system, we must expect boundary layer behavior near each endpoint (cf. Harris (1973) and Ferguson (1975) for discussions of problems where $f_y(y, t, 0) \equiv 0$).

2. Problems with boundary layers at both endpoints

Let us now consider the "twin" boundary layer problem

$$(23) \quad \left\{ \begin{array}{l} \epsilon y'' + g(y, t, \epsilon) = 0, \quad 0 \leq t \leq 1 \\ y(0), y(1) \text{ prescribed} \end{array} \right.$$

under the assumption that g is infinitely differentiable in the region \bar{D} of interest and that the reduced system

$$(24) \quad g(u, t, 0) = 0$$

has a smooth solution $U_0(t)$ throughout $0 \leq t \leq 1$ which satisfies the stability assumption

$$(25) \quad g_y(U_0(t), t, 0) < 0$$

there, i.e., g_y is a stable matrix when evaluated along $(U_0(t), t, 0)$, $0 \leq t \leq 1$. Motivation for this assumption is obvious if one considers the linear scalar problems with $\epsilon y'' + y = 0$, while generalized stability assumptions are sometimes appropriate and necessary (cf., e.g., Howes (1978) or consider the scalar problem with $g = y^{2q+1}$). With (25), one can hope that a solution to (23) exists which converges to $U_0(t)$ within $(0, 1)$. Since we won't generally have either

$U_0(0) = y(0)$ or $U_0(1) = y(1)$, we must expect "twin" endpoint boundary layers (i.e., regions of nonuniform convergence of thickness $0(\sqrt{\epsilon})$ near both $t = 0$ and $t = 1$). Our previous experience (cf. Fife (1973, 1976), Yarmish (1975), O'Malley (1976), and Howes (1978)) suggests that we must add "boundary layer stability" assumptions. These generally limit the size of the boundary layer jumps $\|y(0) - U_0(0)\|$ and $\|y(1) - U_0(1)\|$. They'll certainly be guaranteed if g_y remains stable throughout the boundary layer regions (cf. Kelley (1978)). Indeed, for small boundary layer jumps, the stability assumption (25) is sufficient. Under appropriate assumptions, then, we can expect to obtain an asymptotic solution to (23) in the form

$$(26) \quad y(t, \epsilon) = \tilde{U}(t, \epsilon) + \tilde{v}(\rho, \sqrt{\epsilon}) + \tilde{w}(\sigma, \sqrt{\epsilon})$$

where \tilde{U} , \tilde{v} , and \tilde{w} all have power series expansions in their second variables and the terms of the left boundary layer correction \tilde{v} tend to zero as the stretched variable

$$(27) \quad \rho = t/\sqrt{\epsilon}$$

tends to infinity while the right boundary layer correction $\tilde{w} \rightarrow 0$ as

$$(28) \quad \sigma = (1 - t)/\sqrt{\epsilon}$$

becomes unbounded.

The outer expansion

$$(29) \quad U(t, \epsilon) \sim \sum_{j=0}^{\infty} U_j(t) \epsilon^j$$

should therefore satisfy

$$(30) \quad \epsilon U'' + g(U, t, \epsilon) = 0, \quad 0 < t < 1$$

as a power series in ϵ and converge to the solution $U_0(t)$ of the reduced system (24) as $\epsilon \rightarrow 0$. Higher order terms in (29) must satisfy linear systems of the form

$$(31) \quad g_y(U_0, t, 0) U_j = c_{j-1}(t), \quad j \geq 1$$

where c_{j-1} is known termwise (e.g., $c_0 = -U_0''$). The stability condition (25) implies that the systems (31) are all nonsingular. Therefore successive coefficients are simply and uniquely obtained termwise. (Different roots U_0 of (24)

would, of course, result in different sequences of perturbation terms U_j , $j > 0$, under appropriate stability assumptions.)

According to the Borel-Ritt Theorem (cf. Wasow (1965)), there is a (non-unique) function $\tilde{U}(t, \varepsilon)$, holomorphic in ε , having the outer expansion (29). If we set

$$(32) \quad y(t, \varepsilon) = \tilde{U}(t, \varepsilon) + z(t, \varepsilon),$$

we convert the problem (23) into the two-point problem

$$(33) \quad \varepsilon z'' = h(z, t, \varepsilon), \quad 0 \leq t \leq 1, \quad z(0, \varepsilon) = y(0) - \tilde{U}(0, \varepsilon).$$

Here

$$h(z, t, \varepsilon) = -\varepsilon \tilde{U}''(t, \varepsilon) - g(\tilde{U}(t, \varepsilon) + z, t, \varepsilon)$$

satisfies

$$(34) \quad h(0, t, \varepsilon) = 0(\varepsilon^N) \quad \text{for every integer } N \geq 0$$

since $\varepsilon \tilde{U}'' + g(\tilde{U}, t, \varepsilon) = 0(\varepsilon^N)$. In particular, the reduced system

$$h(z, t, 0) = 0$$

corresponding to the transformed problem (33) has the (not necessarily unique) trivial solution and the outer expansion for (33) is also trivial. Henceforth, then, we shall deal with (33) and, corresponding to (26), we shall seek an asymptotic solution of the form

$$(35) \quad z(t, \varepsilon) = v(\rho, \sqrt{\varepsilon}) + w(\sigma, \sqrt{\varepsilon})$$

providing the needed boundary layer decay to zero within $0 < t < 1$. Our smoothness assumptions will be required in a domain

$$\mathcal{D}_{\varepsilon_1, \delta} = \{(z, t, \varepsilon) : 0 \leq \|z - U_0(t)\| \leq d_\delta(t), \quad 0 \leq t \leq 1, \quad 0 \leq \varepsilon \leq \varepsilon_1\}$$

where ε_1 is a small positive number and, for any $\delta > 0$, we define

$$d_\delta(t) = \begin{cases} \|z(0) - U_0(0)\| + \delta, & 0 \leq t < \delta \\ \delta, & \delta \leq t \leq 1 - \delta \end{cases}$$

$$\left\{ \begin{array}{l} \|z(1) - u_0(1)\| + \delta, \quad 1 - \delta < t \leq 1. \end{array} \right.$$

We shall determine the asymptotic behavior of z by first determining that of $\|z\| = \sqrt{z^T z}$. Here $\|z\|$ satisfies the scalar problem

$$(36) \quad \epsilon \|z\|^n = [h^T(z, t, \epsilon)z + \epsilon(\|z\|^2 - (\|z\|^2)')]/\|z\|,$$

$0 \leq t \leq 1$, where $\|z(0, \epsilon)\|$ and $\|z(1, \epsilon)\|$ are prescribed.

This follows via simple calculations, namely

$$\frac{d}{dt} \|z\|^2 = 2\|z\| \|z\|' = 2(z')^T z$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \|z\|^2 &= 2\|z\| \|z\|'' + 2(\|z\|')^2 \\ &= 2(z'')^T z + 2\|z'\|^2 \end{aligned}$$

imply the differential equation for $\|z\|$. Further,

$$(37) \quad \|z'\|^2 \geq (\|z\|')^2$$

since the Cauchy-Schwarz inequality $((z')^T z)^2 \leq \|z'\|^2 \|z\|^2$ implies that $\|z'\|^2 \geq ((z')^T z / \|z\|)^2 = (\|z\|')^2$. Thus, with a loss whenever z and z' are not collinear,

$$(38) \quad \epsilon \|z\|^n \geq h^T(z, t, \epsilon)z / \|z\|, \quad 0 \leq t \leq 1.$$

(Through the inequality (37), then, we eliminate the first derivative term from (38). We note that (38) is an equality for scalar problems.)

We'll now ask that for all (z, t, ϵ) in $\tilde{D}_{\epsilon_1, \delta}$, there exists a smooth scalar function

$$\phi(n, t, \epsilon)$$

such that

$$(39) \quad h^T(z, t, \epsilon)z \geq \phi(\|z\|, t, \epsilon)\|z\|$$

where

$$(40) \quad \phi(0, t, \varepsilon) \leq 0, \quad \phi(0, t, 0) = 0, \quad \frac{\partial \phi}{\partial n}(0, t, 0) > 0$$

and

$$(41) \quad \left\{ \begin{array}{l} \int_0^n \phi(s, 0, 0) ds > 0 \text{ whenever } 0 < n \leq \|z(0, 0)\| \text{ if } z(0, 0) \neq 0 \\ \text{and} \\ \int_0^{\zeta} \phi(s, 1, 0) ds > 0 \text{ whenever } 0 < \zeta \leq \|z(1, 0)\| \text{ if } z(1, 0) \neq 0. \end{array} \right.$$

Existence of such a function ϕ will constitute our stability hypotheses. Specifically, $\frac{\partial \phi}{\partial n}(0, t, 0) > 0$ implies the stability of the trivial solution of the reduced system within $(0, 1)$ while (41) implies boundary layer stability at both endpoints. Hypotheses (39)-(40) imply that

$$(42) \quad 0 \leq \|z(t, \varepsilon)\| \leq m(t, \varepsilon)$$

where $m(t, \varepsilon)$ satisfies the scalar two-point problem

$$(43) \quad \varepsilon m'' = \phi(m, t, \varepsilon), \quad 0 \leq t \leq 1, \quad m(0, \varepsilon) = \|z(0, \varepsilon)\|, \quad m(1, \varepsilon) = \|z(1, \varepsilon)\|.$$

The bounds (42) follow from the elementary theory of differential inequalities since zero is a lower solution for $\|z\|$ and m is an upper solution (cf. Nagumo (1937), Dorr, Parter, and Shampine (1973), and Howes (1976)). Further, $\phi(0, t, 0) = 0$ and $\frac{\partial \phi}{\partial n}(0, t, 0) > 0$ imply that the zero solution of the reduced problem $\phi(m, t, 0) = 0$ corresponding to (43) is stable and, according to Howes (1978), (41) is the appropriate hypothesis for the needed boundary layer stability of this solution. Indeed, the solution of (43) satisfies

$$(44) \quad m(t, \varepsilon) = r_0(\rho) + s_0(\sigma) + o(\sqrt{\varepsilon})$$

where r_0 is the decaying solution of the boundary layer problem

$$(45) \quad \frac{d^2 r_0}{d\rho^2} = \phi(r_0, 0, 0), \quad \rho \geq 0, \quad r_0(0) = \|z(0, 0)\| = \|y(0) - u_0(0)\|$$

while s_0 is the decaying solution of

$$(46) \quad \frac{d^2 s_0}{d\sigma^2} = \phi(s_0, 1, 0), \quad \sigma \geq 0, \quad s_0(0) = \|z(1, 0)\| = \|y(1) - u_0(1)\|.$$

The solutions to (45) and (46) are easily shown to exist and be unique. Multiplying (45) by $dr_0/d\rho$, for example, and integrating from ρ to infinity implies that

$$\frac{1}{2} \left(\frac{dr_0}{d\rho} \right)^2 = \int_0^{r_0(\rho)} \phi(s, 0, 0) ds > 0$$

(by (41)). Thus, r_0 satisfies the initial value problem

$$(47) \quad \frac{dr_0}{d\rho} = - \sqrt{2 \int_0^{r_0(\rho)} \phi(s, 0, 0) ds}, \quad r_0(0) = \|y(0) - u_0(0)\|.$$

Hence, $r_0(\rho)$ will decrease monotonically to zero as ρ increases, reaching the rest point $r_0 = 0$ at $\rho = \infty$. Since $\phi(s, 0, 0) \approx \frac{\partial \phi}{\partial n}(0, 0, 0)s$ for s small, $\frac{\partial \phi}{\partial n}(0, 0, 0) > 0$ implies that the decay of r_0 to zero is exponential as $\rho \rightarrow \infty$. (When $r_0(0) = 0$, we have $r_0(\rho) \equiv 0$ since there is no need for a boundary layer correction.) Continuing by solving linear problems, we could obtain an asymptotic solution of (43) in the form

$$(48) \quad m(t, \varepsilon) = r(\rho, \sqrt{\varepsilon}) + s(\sigma, \sqrt{\varepsilon}).$$

In terms of the original problem (23), our stability hypothesis (39) becomes the inequality

$$g^T(u_0(0) + z, t, \varepsilon)z \leq \phi(\|z\|, t, \varepsilon)\|z\|$$

where ϕ satisfies (40) and (41). The expansion (44) corresponds to the expected expansion (26) for an asymptotic solution for the vector problem (23).

Now, we return to the vector boundary value problem (33) and its asymptotic solution in the form (35). Near $t = 0$, w and its derivatives should be asymptotically negligible (σ being infinite), so (33) and (35) imply that the initial boundary layer correction v should be a decaying solution of the nonlinear initial value problem

$$(49) \quad v_{\rho\rho} = h(v, \sqrt{\varepsilon}\rho, \varepsilon), \quad \rho \geq 0, \quad v(0, \varepsilon) = z(0, \varepsilon).$$

Thus, it is natural to seek an expansion

$$(50) \quad v(\rho, \sqrt{\varepsilon}) \sim \sum_{j=0}^{\infty} v_j(\rho) \varepsilon^{j/2}$$

by substitution into (49). The leading term v_0 must then satisfy the nonlinear

problem

$$(51) \quad \frac{d^2 v_0}{d\rho^2} = h(v_0, 0, 0), \quad \rho \geq 0, \quad v_0(0) = y(0) - u_0(0), \quad v_0 \rightarrow 0 \text{ as } \rho \rightarrow \infty.$$

Later terms v_j , $j \geq 1$, must satisfy the linear problems

$$(52) \quad \left\{ \begin{array}{l} \frac{d^2 v_j}{d\rho^2} = h_z(v_0, 0, 0)v_j + d_{j-1}(\rho), \quad \rho \geq 0 \\ v_j(0) = 0, \quad j \text{ odd}; \quad v_j(0) = -u_{j/2}(0), \quad j \text{ even} \\ v_j \rightarrow 0 \text{ as } \rho \rightarrow \infty \end{array} \right.$$

where d_{j-1} will be determined successively as an exponentially decaying vector. Since (51) and our hypothesis (39) imply that

$$\|v_0\|_{\rho\rho} \geq \phi(\|v_0\|, 0, 0), \quad \rho \geq 0, \quad \|v_0(0)\| = r_0(0),$$

we are guaranteed a decaying solution $v_0(\rho)$ such that

$$(53) \quad 0 \leq \|v_0(\rho)\| \leq r_0(\rho), \quad \rho \geq 0$$

(and $v_0(\rho) \equiv 0$ if $u_0(0) = y(0)$). No explicit solution v_0 can be provided, though an approximate solution can be obtained as usual. Introducing the matrix

$$\alpha = h_z(0, 0, 0) > 0$$

(whose eigenvalues have strictly positive real parts by our stability assumption (25)), variation of parameters can be used to express the solution of (52) in the form

$$(54) \quad v_j(\rho) = e^{-\sqrt{\alpha}\rho} \left[v_j(0) - \int_0^\infty e^{-\sqrt{\alpha}s} F_j(s) ds \right] - \frac{1}{2} (\sqrt{\alpha})^{-1} \left[\int_0^\rho e^{-\sqrt{\alpha}(\rho-r)} F_j(r) dr + \int_\rho^\infty e^{\sqrt{\alpha}(s-\rho)} F_j(s) ds \right]$$

where $F_j(\rho) = [h_z(v_0(\rho), 0, 0) - \alpha]v_j(\rho) + d_{j-1}(\rho)$. This provides the exact solution to (52) whenever $h(v_0, 0, 0)$ is linear. Otherwise, the linear integral equation (54) must also be solved by successive approximations. In analogous fashion, we could generate the terms of the terminal boundary layer correction $w(\sigma, \sqrt{\epsilon})$ of (35). Thus, we've formally obtained (35), which we expect is a locally unique asymptotic solution.

We note that the assumptions on ϕ automatically hold if $g_y(y, t, 0)$ or $-h_z(z, t, 0)$ are everywhere stable. Thus, if we take

$$h_z(v, 0, 0) - \gamma I > 0$$

(i.e., positive definite) for some real $\gamma > 0$ and all v satisfying $0 \leq \|v\| \leq \|z(0, 0)\|$, the mean value theorem implies that

$$h^T(z, 0, 0)z = z^T h_z(\tilde{z}, 0, 0)z > \gamma \|z\|^2$$

for some "intermediate" point \tilde{z} . Thus, taking $\phi(n, 0, 0) = \gamma n$, both $\frac{\partial \phi}{\partial n}(0, 0, 0) > 0$ and $\int_0^n \phi(s, 0, 0)ds > 0$ for $0 < n \leq \|z(0, 0)\|$ hold.

We could also extend our discussion to systems of the form

$$\epsilon x'' = F(x, x', t, \epsilon)$$

with $\frac{\partial F}{\partial x'}$ small. Thus, Kelley (1978) considered problems where $\frac{\partial F}{\partial x} - \frac{1}{2\epsilon} \frac{\partial F}{\partial x'} \left(\frac{\partial F}{\partial x'} \right)^T > 0$, just as Erdélyi (1968) considered scalar problems somewhat more nonlinear than semilinear.

3. Examples

a. A problem with an initial boundary layer

Let us consider the vector equation

$$\epsilon y'' + f(y, t, \epsilon)y' + g(y, t, \epsilon) = 0, \quad 0 \leq t \leq 1$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad f = \begin{pmatrix} y_1 & 1 \\ 1 & y_2 \end{pmatrix}, \quad \text{and} \quad g = -\begin{pmatrix} y_1 + 1 \\ y_2 + 1 \end{pmatrix}.$$

In order to have a limiting solution u_R of the two-point problem which satisfies the reduced problem

$$f(u_R, t, 0)u_R' + g(u_R, t, 0) = 0, \quad u_R(1) = y(1)$$

we must require u_R to be stable in $0 \leq t \leq 1$, i.e.,

$$-f(u_R(t), t, 0) < 0$$

must be a stable matrix, and we must also require boundary layer stability at

$t = 0$, i.e., we ask that

$$\zeta^T \int_0^\zeta f(u_R(0) + z, 0, 0) dz > 0$$

for all ζ such that $0 < \|\zeta\| \leq \|y(0) - u_R(0)\|$.

More specifically, the reduced problem has the solution

$$u_R(t) = \begin{pmatrix} t + C \\ t + D \end{pmatrix}$$

where $C = u_{R1}(0) = -1 + y_1(1)$ and $D = u_{R2}(0) = -1 + y_2(1)$. Stability of u_R requires the matrix

$$\begin{pmatrix} -t - C & -1 \\ -1 & -t - D \end{pmatrix}$$

to be stable throughout $0 \leq t \leq 1$. This is, however, equivalent to asking that

$$C + D > 0 \quad \text{and} \quad CD > 1,$$

i.e.,

$$y_1(1)y_2(1) > y_1(1) + y_2(1) > 2.$$

Further, boundary layer stability requires that

$$\zeta^T \int_0^\zeta \begin{pmatrix} w_1 + C & 1 \\ 1 & w_2 + D \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} > 0,$$

i.e.,

$$\zeta_1^3 + 2C\zeta_1^2 + 4\zeta_1\zeta_2 + \zeta_2^3 + 2D\zeta_2^2 > 0$$

for all $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ satisfying $0 < \|\zeta\| = \|y(0) - u_R(0)\| = \sqrt{(y_1(0) - C)^2 + (y_2(0) - D)^2}$. Our initial values $y(0)$ are thereby restricted to a circle about (C, D) with radius less than the least norm $\|\zeta\|$ of the nontrivial zeros of the cubic polynomial. Setting $\zeta_2 = t\zeta_1$, such a ζ will satisfy

$$(1 + t^3)\zeta_1 = -2(C + 2t + Dt^2)$$

and we minimize

$$d(t) = \|\zeta\| = \sqrt{1 + t^2} |\zeta_1|.$$

(We note that the minimum for $\zeta_1 = 0$, $t = \infty$, is 2D.) This calculus problem, then, determines an upper bound for $\|y(0) - u_R(0)\|$.

For $C = D = 2$, i.e., $y(1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, we'd obtain the minimum value 3.390 for $d(t)$ corresponding to $t_{\min} = -0.291$. Thus, we're guaranteed that the limiting solution of our two-point problem is provided by $u_R(t)$ if $y(0)$ lies in the circle of radius 3.390 about $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. This is presumably a conservative estimate for the "domain of attraction" of the reduced solution $u_R(t)$. We expect that boundary layer stability need only hold for $\zeta + u_R(0)$ on the actual trajectory joining $y(0)$ and $u_R(0)$. Finally, we observe that this example is quite analogous to the simplest cases occurring in the analysis of solutions of the scalar problem $\epsilon y'' + yy' - y = 0$ (cf. Cole (1968), Howes (1978), and elsewhere).

b. A problem with twin boundary layers at the endpoints

Consider the vector problem

$$\epsilon z'' = h(z, t, \epsilon), \quad 0 \leq t \leq 1$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} z_1 + z_2 - z_1^3 \\ -z_1 + z_2 - z_2^3 \end{pmatrix}.$$

Here $U_0 = 0$ is a stable solution of the reduced problem $h(U_0, t, 0) = 0$ since the Jacobian matrix

$$h_z(0, t, 0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has the unstable eigenvalues $1 \pm i$. Boundary layer stability involves the determination of a scalar function ϕ such that

$$h^T(z, t, \epsilon)z \geq \phi(\|z\|, t, \epsilon)\|z\|.$$

Here

$$h^T(z, t, \epsilon)z = (z_1^2 + z_2^2) - (z_1^4 + z_2^4) \geq \|z\|^2(1 - \|z\|^2).$$

Since $z_1^4 + z_2^4 \leq (z_1^2 + z_2^2)^2$, so we can take

$$\phi(n, t, \epsilon) = n(1 - n^2).$$

Clearly, $\phi(0, t, \epsilon) \leq 0$, $\phi(0, t, 0) = 0$, $\phi_n(0, t, 0) > 0$ and

$$\int_0^n \phi(s, i, 0) ds = \frac{1}{2} n^2 (1 - n^2/2) > 0 \quad \text{for } 0 < n < \sqrt{2}, \quad i = 0 \text{ or } 1.$$

Our preceding results, then, guarantee the existence of an asymptotic solution to the two-point problem which converges to the limiting solution $U_0 = 0$ within $(0, 1)$ provided the boundary values satisfy

$$\|z(0, 0)\| < \sqrt{2} \quad \text{and} \quad \|z(1, 0)\| < \sqrt{2}.$$

Indeed, we then have

$$0 \leq \|z(t, \epsilon)\| \leq m(t, \epsilon)$$

where m satisfies the scalar problem

$$\epsilon m'' = \phi(m, t, \epsilon), \quad 0 \leq t \leq 1, \quad m(i, \epsilon) = \|z(i, \epsilon)\| < \sqrt{2}, \quad i = 0 \text{ and } 1.$$

The asymptotic behavior of m follows from the scalar results of Howes (1978) and others.

c. A problem with internal transition layers

We now consider the very special problem

$$\epsilon y'' + f(y, t, \epsilon) y' + g(y, t, \epsilon) = 0, \quad 0 \leq t \leq 1$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad f(y, t, \epsilon) = \begin{pmatrix} f_1(y_1, t, \epsilon) & f_2(y_1, y_2, t, \epsilon) \\ 0 & y_2 \end{pmatrix},$$

and $g = \begin{pmatrix} g_1(y_1, y_2) \\ -y_2 \end{pmatrix}.$

This system decouples into the two nonlinear scalar equations

$$\epsilon y_2'' + y_2 y_2' - y_2 = 0$$

and

$$\epsilon y_1'' + f_1(y_1, t, \epsilon) y_1' + [f_2(y_1, y_2, t, \epsilon) y_2' + g_1(y_1, y_2, t, \epsilon)] = 0.$$

If $y_2(1) > y_2(0) + 1$ and $-y_2(1) - 1 < y_2(0) < 1 - y_2(1)$, it follows from Howes (1978) that the limiting solution for y_2 will satisfy the reduced problem

$$u_L(u_L' - 1) = 0, \quad u_L(0) = y_2(0) \quad \text{on} \quad 0 \leq t < t^* = \frac{1}{2} (1 - y_2(1) - y_2(0))$$

and the reduced problem

$$u_R(u_R' - 1) = 0, \quad u_R(1) = y_2(1) \quad \text{on} \quad t^* < t \leq 1,$$

i.e.,

$$y_2 \rightarrow u = \begin{cases} u_L(t) = t + y_2(0), & 0 \leq t < t^* \\ u_R(t) = t + y_2(1) - 1, & t^* < t \leq 1. \end{cases}$$

Thus, the limiting solution is generally discontinuous at t^* and its derivative (which is asymptotically one elsewhere) becomes unbounded there. Indeed, y_2 increases monotonically near t^* from $u_L(t^*)$ to $u_R(t^*) = -u_L(t^*)$. For other relations between the boundary values $y_2(0)$ and $y_2(1)$, other limiting possibilities occur (cf., e.g., Howes).

One must generally expect the transition layer at t^* in y_2 to generate a corresponding discontinuity there in y_1 . To simplify our discussion, however, let's assume that $f_2(y_1, y_2, t, 0) = 0$ and attempt to apply Howes' scalar theory to the equation for y_1 . Thus, consider the reduced problems

$$f_1(v_L, t, 0)v_L' + g_1(v_L, u, t, 0) = 0, \quad 0 \leq t \leq 1, \quad v_L(0) = y_1(0)$$

and

$$f_1(v_R, t, 0)v_R' + g_1(v_R, u, t, 0) = 0, \quad 0 \leq t \leq 1, \quad v_R(1) = y_1(1).$$

The limiting solution for y_1 will be provided by $v_R(t)$ if the stability condition

$$f_1(v_R(t), t, 0) > 0$$

holds throughout $0 \leq t \leq 1$ and the boundary layer stability assumption

$$(v_R(0) - y_1(0)) \int_0^{v_R(0)} f_1(s, 0, 0) ds > 0$$

for n between $v_R(0)$ and (including) $y_1(0)$. Similar conditions would imply that the limiting solution is $v_L(t)$ on $0 \leq t < 1$ with boundary layer behavior

near $t = 1$. If, instead, we have

$$f_1(v_R(t), t, 0) > 0 \quad \text{on} \quad t_R \leq t \leq 1$$

while

$$f_1(v_L(t), t, 0) < 0 \quad \text{on} \quad 0 \leq t \leq t_L$$

with $t_R < t_L$, we can expect y_1 to have a limiting solution

$$y_1 \rightarrow v = \begin{cases} v_L(t), & 0 \leq t < \hat{t} \\ v_R(t), & \hat{t} < t \leq 1 \end{cases}$$

as $\epsilon \rightarrow 0$ provided we can find a \hat{t} in (t_R, t_L) such that

$$J(\hat{t}) = 0, \quad J'(\hat{t}) \neq 0$$

for

$$J(t) = \int_{v_L(t)}^{v_R(t)} f_1(s, t, 0) ds$$

(cf. Howes (1978)). Pictorially, we will have limiting solutions y_2 and y_1 as shown in Figures 2 and 3.

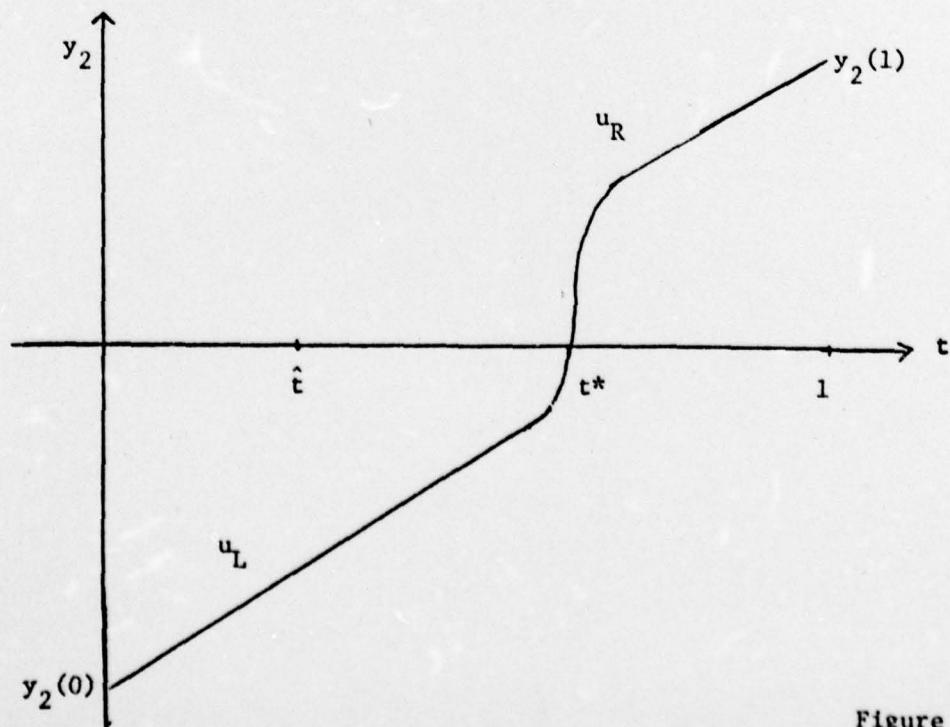


Figure 2

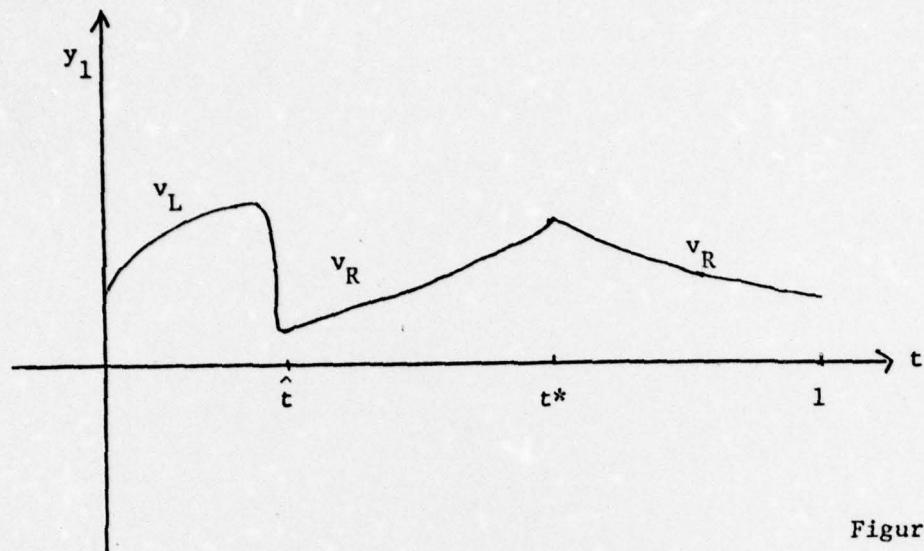


Figure 3

Note that y_2 has a jump at \hat{t} and y_2' has a jump at t^* , corresponding to a Haber-Levinson crossing (cf. Howes (1978)). Much more complicated possibilities remain to be studied.

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